Pinch instabilities in Taylor-Couette flow

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The linear stability of the dissipative Taylor-Couette flow with an azimuthal magnetic field is considered. Unlike ideal flows, the magnetic field is a fixed function of a radius with two parameters only: a ratio of inner to outer cylinder radii, $\hat{\eta}$, and a ratio of the magnetic field values on outer and inner cylinders, $\hat{\mu}_B$. The magnetic field with $0<\hat{\mu}_B<1/\hat{\eta}$ stabilizes the flow and is called a stable magnetic field. The current free magnetic field ($\hat{\mu}_B=\hat{\eta}$) is the stable magnetic field. The unstable magnetic field, which value (or Hartmann number) exceeds some critical value, destabilizes every flow including flows which are stable without the magnetic field. This instability survives even without rotation. The unstable modes are located into some interval of the axial wave numbers for the flow stable without magnetic field. The interval length is zero for a critical Hartmann number and increases with an increasing Hartmann number. The critical Hartmann numbers and length of the unstable axial wave number intervals are the same for every rotation law. There are the critical Hartmann numbers for $m=0$ sausage and $m=1$ kink modes only. The sausage mode is the most unstable mode close to Ha=0 point and the kink mode is the most unstable mode close to the critical Hartmann number. The transition from the sausage instability to the kink instability depends on the Prandtl number Pm and this happens close to one-half of the critical Hartmann number for Pm=1 and close to the critical Hartmann number for Pm=10^{-5}. The critical Hartmann numbers are smaller for kink modes. The flow stability does not depend on magnetic Prandtl numbers for $m=0$ mode. The same is true for critical Hartmann numbers for both $m=0$ and $m=1$ modes. The typical value of the magnetic field destabilizing the liquid metal Taylor-Couette flow is $\sim 10^2$ G.

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I. INTRODUCTION

The Taylor-Couette flow between concentric rotating cylinders is a classical problem of hydrodynamic and hydromagnetic stability, [1,2]. According to the Rayleigh criterion the ideal flow is stable whenever the specific angular momentum increases outwards

$$\frac{d}{dR}(R^2\Omega)^2 > 0, \tag{1}$$

where the cylindrical system of coordinate (R,ϕ,z) is used, and Ω is the angular velocity.

The axial magnetic field destabilizes the ideal flow stable according to (1) but with angular velocity decreasing outwards and changes the stability condition to

$$\frac{d\Omega^2}{dR} > 0. \tag{2}$$

This magnetorotational instability (MRI) has been discovered decades ago [3], but its importance as the source of turbulence in accretion disks with differential (Keplerian) rotation was only recognized much latter by Balbus and Hawley [4]. Their local stability analysis suggests instability regardless of the magnitude of the azimuthal magnetic field. It was not a surprise that this result has been reconsidered later [5–7] in light of the long-time known Michael's necessary and sufficient condition [8]. The condition says that the ideal Taylor-Couette flow is stable to axisymmetric disturbances in the presence of an azimuthal magnetic field $B_\phi(R)$ if

$$\frac{1}{R^3}\frac{d}{dR}(R^2\Omega)^2 - \frac{R}{\mu_0\rho}\frac{d}{dR}\left(\frac{B_\phi}{R}\right)^2 > 0, \tag{3}$$

where ρ is the density and $\mu_0=4\pi$ is the magnetic constant. According to (3), a Taylor-Couette flow with an arbitrary angular velocity profile is unstable to axisymmetric disturbances for appropriate azimuthal magnetic field values and profiles. The destabilizing role of the azimuthal magnetic field is also well known in the plasma theory of pinch stability (see, e.g., [9]).

The viscosity has a stabilizing effect and a nonmagnetized Taylor-Couette flow which is unstable due to (1) becomes really unstable only if the angular velocity of inner cylinder (or its Reynolds number) exceeds some critical value. The same is true for the nonideal Taylor-Couette flow with an imposed axial magnetic field. Moreover, MRI exists in hydrodynamically unstable situations ($\hat{\mu}_\Omega<\hat{\eta}^2$) [24] only if the magnetic Prandtl number Pm is not very small as shown in [10] already and later in [11–14]; the critical Reynolds numbers vary as 1/Pm for hydrodynamically stable flows ($\hat{\eta}^2<\hat{\mu}_\Omega<1$) [11,14] so that it is the magnetic Reynolds number which directs the instability. Pm is really very small for liquid metals (10^{-5} and smaller). That is why the MRI has not been clearly demonstrated experimentally. MRI-like behavior was reported at recent experiments [15]. Nevertheless, the initial nonmagnetized flow was already unstable (turbulent) and a relation of these results with the MRI of the laminar flow is not clear.

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The importance of the MRI for accretion disk physics and for planned new experiments [13,16,17] highly stimulated the theoretical investigation of the stability of the magnetized Taylor-Couette flow [11–14,16–20].

Theoretical results for the viscous magnetized Taylor-Couette flow with an imposed azimuthal magnetic field is not so numerous. The only work [21], has demonstrated flow stabilization only. This work was restricted by a current-free azimuthal magnetic field and conducting boundary conditions. Here we perform a more comprehensive study of the nonideal Taylor-Couette flow stability in the presence of the azimuthal magnetic field. We consider the most general configuration of the azimuthal magnetic field and as conducting so insulating boundary conditions.

II. BASIC EQUATIONS

Consider a viscous electrically conducting incompressible fluid between two rotating infinite cylinders in the presence of an azimuthal magnetic field. The equations governing the problem are

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\,\boldsymbol{\nabla})\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\Delta\mathbf{U} + \frac{1}{\mu_0}\mathrm{curl}\mathbf{B}\times\mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{U}\times\mathbf{B}) + \eta\Delta\mathbf{B},$$

$$\mathrm{div}\,\mathbf{U} = \mathrm{div}\,\mathbf{B} = 0, \tag{4}$$

where $\mathbf{U}$ is the velocity, $\mathbf{B}$ is the magnetic field, P is the pressure, ν is the kinematic viscosity, and η is the magnetic diffusivity. Equations (4) admit the basic solution in the cylindrical system of coordinates (R,ϕ,z)

$$U_R = U_z = B_R = B_z = 0,$$

$$B_\phi = a_B R + \frac{b_B}{R}, \quad U_\phi = R\Omega = a_\Omega R + \frac{b_\Omega}{R}, \tag{5}$$

a_Ω, b_Ω, a_B, and b_B are constants defined by boundary conditions:

$$a_\Omega = \Omega_{\mathrm{in}}\frac{\hat{\mu}_\Omega - \hat{\eta}^2}{1-\hat{\eta}^2}, \quad b_\Omega = \Omega_{\mathrm{in}}R_{\mathrm{in}}^2\frac{1-\hat{\mu}_\Omega}{1-\hat{\eta}^2},$$

$$a_B = \frac{B_{\mathrm{in}}}{R_{\mathrm{in}}}\frac{\hat{\eta}(\hat{\mu}_B - \hat{\eta})}{1-\hat{\eta}^2}, \quad b_B = B_{\mathrm{in}}R_{\mathrm{in}}\frac{1-\hat{\mu}_B\hat{\eta}}{1-\hat{\eta}^2}, \tag{6}$$

where

$$\hat{\eta} = \frac{R_{\mathrm{in}}}{R_{\mathrm{out}}}, \quad \hat{\mu}_\Omega = \frac{\Omega_{\mathrm{out}}}{\Omega_{\mathrm{in}}}, \quad \hat{\mu}_B = \frac{B_{\mathrm{out}}}{B_{\mathrm{in}}}, \tag{7}$$

R_{in} and R_{out} are the radii, Ω_{in} and Ω_{out} are the angular velocities, and B_{in} and B_{out} are the azimuthal magnetic fields of the inner and the outer cylinders, respectively.

Note, that for the viscous flow the magnetic field profile like the angular velocity profile is a fixed function of the radius. The first magnetic field term at (5) corresponds to a constant axial electric current density into the fluid. The second term is current free.

We are interested in the stability of the basic solution (5). The linear stability problem is considered. The perturbed state of the flow is described by

$$u_R,\ R\Omega + u_\phi,\ u_z,\ b_R,\ B_\phi + b_\phi,\ b_z. \tag{8}$$

By developing the disturbances into normal modes, the solutions of the linearized magnetohydrodynamics (MHD) equations are considered in the form

$$F = F(R)\exp[i(kz + m\phi + \omega t)], \tag{9}$$

where F is every velocity or magnetic field disturbances.

The dimensionless numbers of the problem are the magnetic Prandtl number, Pm, Hartmann number, Ha, and Reynolds number, Re,

$$\mathrm{Pm} = \frac{\nu}{\eta}, \quad \mathrm{Ha} = \frac{B_{\mathrm{in}}R_0}{\sqrt{\mu_0\rho\nu\eta}}, \quad \mathrm{Re} = \frac{\Omega_{\mathrm{in}}R_0^2}{\nu}, \tag{10}$$

where $R_0 = [R_{\mathrm{in}}(R_{\mathrm{out}} - R_{\mathrm{in}})]^{1/2}$ is the length unit.

Using normal mode expansion (9), linearizing system (4) and representing it as a system of first order equations, we have

$$\frac{du_R}{dR} + \frac{u_R}{R} + i\frac{m}{R}u_\phi + iku_z = 0,$$

$$\frac{du_\phi}{dR} + \frac{u_\phi}{R} - X_2 = 0,$$

$$\frac{du_z}{dR} - X_3 = 0,$$

$$\frac{dP}{dR} + i\frac{m}{R}X_2 + ikX_3 + \left(k^2 + \frac{m^2}{R^2}\right)u_R + i\,\mathrm{Re}(\omega + m\Omega)u_R - 2\Omega\,\mathrm{Re}\,u_\phi - i\,\mathrm{Ha}^2\frac{m}{R}B_\phi b_R + 2\,\mathrm{Ha}^2\frac{B_\phi}{R}b_\phi = 0,$$

$$\frac{dX_2}{dR} - \left(k^2 + \frac{m^2}{R^2}\right)u_\phi - i\,\mathrm{Re}(\omega + m\Omega)u_\phi + 2i\frac{m}{R^2}u_R - \frac{\mathrm{Re}}{R}\frac{d}{dR}(R^2\Omega)u_R + i\,\mathrm{Ha}^2\frac{m}{R}B_\phi b_\phi - i\frac{m}{R}P + \frac{\mathrm{Ha}^2}{R}\frac{d}{dR}(RB_\phi)b_R = 0,$$

$$\frac{dX_3}{dR} + \frac{X_3}{R} - \left(k^2 + \frac{m^2}{R^2}\right)u_z - i\,\mathrm{Re}(\omega + m\Omega)u_z - ikP + i\,\mathrm{Ha}^2\frac{m}{R}B_\phi b_z = 0,$$

$$\frac{db_R}{dR} + \frac{b_R}{R} + i\frac{m}{R}b_\phi + ikb_z = 0,$$

$$\frac{db_\phi}{dR} + \frac{b_\phi}{R} - X_4 = 0,$$

$$\frac{db_z}{dR} - \frac{i}{k}\left(k^2 + \frac{m^2}{R^2}\right)b_R + \mathrm{Pm\,Re}\,\frac{1}{k}(\omega + m\Omega)b_R + \frac{1}{k}\frac{m}{R}X_4 - \frac{1}{k}\frac{m}{R}B_\phi u_R = 0,$$

$$\frac{dX_4}{dR} - \left(k^2 + \frac{m^2}{R^2}\right)b_\phi - i\,\mathrm{Pm\,Re}(\omega + m\Omega)b_\phi + i\frac{2m}{R^2}b_R - R\frac{d}{dR}\left(\frac{B_\phi}{R}\right)u_R + \mathrm{Pm\,Re}\,R\frac{d\Omega}{dR}b_r + i\frac{m}{R}B_\phi u_\phi = 0, \tag{11}$$

where 2nd, 3rd, and 8th equations are the definitions of the X_2, X_3, and X_4, respectively. We use R_0 as unit of a length and R_0^{-1} as unit of the wave number, η/R_0 as unit of the perturbed velocity, Ω_{in} as unit of angular velocity, and ω and B_{in} as units of magnetic fields (basic and disturbed).

An appropriate set of ten boundary conditions is needed to solve the system (11). Always no-slip conditions for the velocity on the walls are used, i.e.,

$$u_R = u_\phi = u_z = 0. \tag{12}$$

The boundary conditions for the magnetic field depend on the electrical properties of the walls. The tangential currents and the radial components of the magnetic field vanish on conducting walls hence

$$db_\phi/dR + b_\phi/R = b_R = 0. \tag{13}$$

These boundary conditions hold both for $R=R_{\mathrm{in}}$ and for $R=R_{\mathrm{out}}$.

The situation changes for insulating walls. The magnetic field must match the external magnetic field for vacuum. The condition $\mathrm{curl}_R\mathbf{B}=0$ in vacuum immediately provides

$$b_\phi = \frac{m}{kR}b_z \tag{14}$$

at $R=R_{\mathrm{in}}$ and $R=R_{\mathrm{out}}$. From the solution of the potential equation $\Delta\psi=0$ (where $\mathbf{B}=\nabla\psi$) one finds

$$b_R + \frac{ib_z}{I_m(kR)}\left(\frac{m}{kR}I_m(kR) + I_{m+1}(kR)\right) = 0 \tag{15}$$

for $R=R_{\mathrm{in}}$ and

$$b_R + \frac{ib_z}{K_m(kR)}\left(\frac{m}{kR}K_m(kR) - K_{m+1}(kR)\right) = 0 \tag{16}$$

for $R=R_{\mathrm{out}}$. I_m and K_m are the modified Bessel functions with finite limits at $R\to 0$ and $R\to\infty$, respectively.

The homogeneous set of equations (11) together with the boundary conditions determine the eigenvalue problem of the form $\mathcal{L}(k,m,\mathcal{R}(\omega),\mathrm{Pm},\mathrm{Re},\mathrm{Ha})=0$ [25], where $\mathcal{R}(\omega)$ is the real part of ω. Generally, $\mathcal{L}$ is a complex quantity. It takes the value zero if and only if all parameters are eigenvalues.

The system (11) is approximated by finite differences with typically 200 radial grid points. We can also stress that the results are numerically robust as an increase of the number of grid points does not change the results remarkably. Both real and imaginary parts of $\mathcal{L}$ equalize to value zero by varying $\mathcal{R}(\omega)$ and Reynolds number values for fixed other parameters. There is always minimum of Re eigenvalues for a certain wave number and $\mathcal{R}(\omega)$. This minimum eigenvalue is the desired Reynolds number and called a critical Reynolds number.

It is well known that for the nonmagnetized Taylor-Couette flow, [23], and the Taylor-Couette flow with an imposed axial magnetic field [22], the axisymmetric instability is monotonic (overstability) with $\mathcal{R}(\omega)=0$. Our calculations (not comprehensive though) have demonstrated that the same is true for the Taylor-Couette flow with azimuthal magnetic field. Thus, for the sake of simplicity, we take that $\mathcal{R}(\omega)=0$ for axisymmetric disturbances below.

III. RESULTS

Using (5)–(7) for angular velocity and magnetic field, the normalized Michael's condition (3) takes the form

$$4a_\Omega^2 + 4\frac{a_\Omega b_\Omega}{R^2} + \alpha\left(4\frac{a_B b_B}{R^2} + 4\frac{b_B^2}{R^4}\right) > 0, \tag{17}$$

where

$$\alpha = \frac{V_{\mathrm{A}}^2}{(R_0\Omega_{\mathrm{in}})^2}, \tag{18}$$

and V_{A} is the Alfvén velocity ($V_{\mathrm{A}}^2=B_{\mathrm{in}}^2/\mu_0\rho$).

According to (1), the angular velocity part [i.e., the sum of the first two terms of (17)] is positive if $\hat{\mu}_\Omega > \hat{\eta}^2$ (see, e.g., [22]). The magnetic field stabilizes the flow (the sum of the last two terms are positive) if

$$0 \leqslant \hat{\mu}_B \leqslant \frac{1}{\hat{\eta}} \tag{19}$$

and destabilize the flow otherwise. The magnetic field is called either stable magnetic field if $\hat{\mu}_B$ lays into interval (19) or the unstable magnetic field otherwise. For the unstable magnetic field there is some critical value of the constant α depending on $\hat{\mu}_\Omega$ for which the ideal flow becomes unstable. Let us note that large values of the constant α (i.e., large magnetic field and slow rotation) are more preferable for instability.

For the nonideal Taylor-Couette flow we are interested in critical Reynolds numbers. Figures 1 and 2 present the critical Reynolds numbers for axisymmetric disturbances as a function of the Hartmann number for insulating and conducting cylinders. The critical Reynolds numbers do not depend on magnetic Prandtl numbers. This result can easy be obtained analytically. For axisymmetric disturbances with $\omega=0$ equations for db_R/dR and db_z/dR can be combined into equation

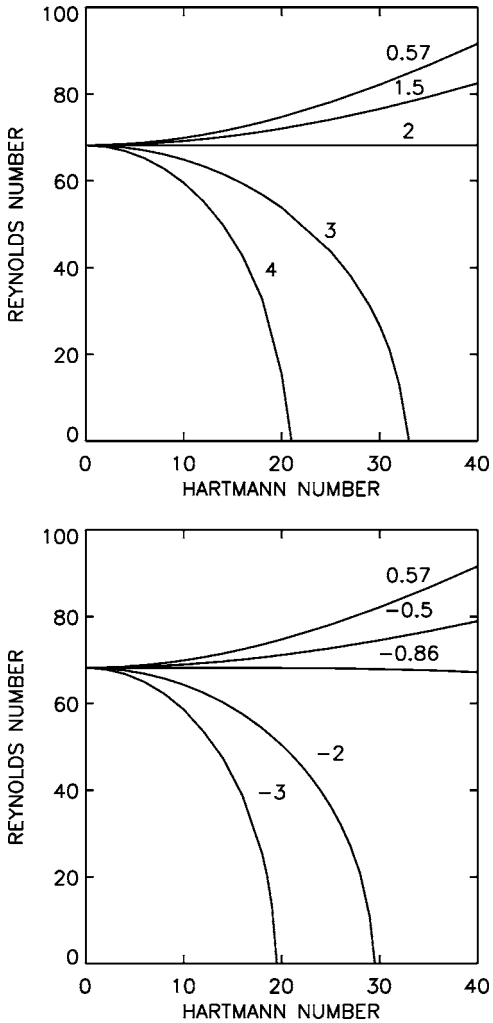

FIG. 1. The marginal stability lines for axisymmetric disturbances ($m=0$) at *insulating* cylinders for $\hat{\eta}=0.5$, $\hat{\mu}_\Omega=0$, $\hat{\mu}_B>0.57$ (upper) and $\hat{\mu}_B<0.57$ (lower). The lines are labeled by the $\hat{\mu}_B$ values.

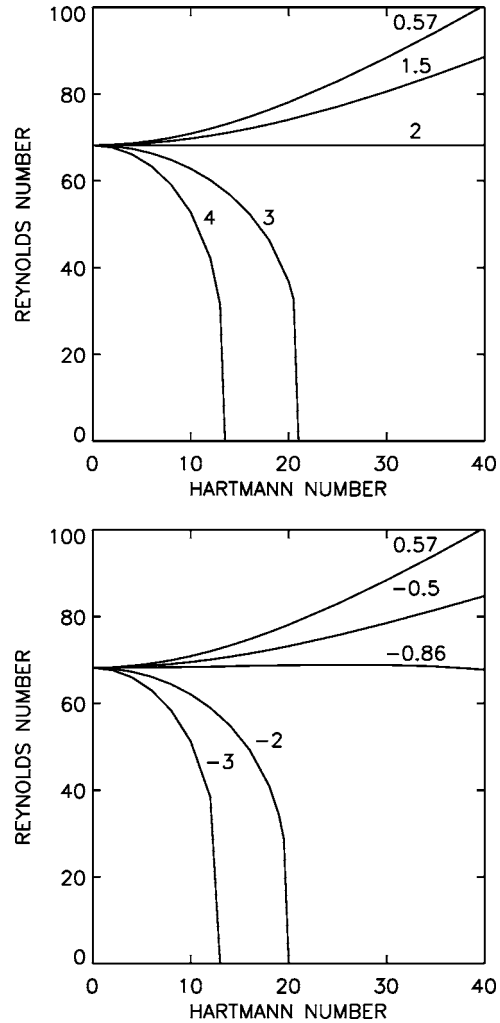

FIG. 2. The same as in Fig. 1 but for *conducting* cylinders.

$$\frac{d^2 b_R}{dR^2} + \frac{1}{R}\frac{db_R}{dR} - \frac{b_R}{R^2} - k^2 b_R = 0. \tag{20}$$

Using (20) together with $m=0$ and $\omega=0$, we can exclude the only term proportional to Pm from the last equation of (11).

After Figs. 1 and 2 it seems that the interval (19) where the magnetic field suppresses the instability for an ideal flow is changed for a nonideal one. The calculations with larger Ha number show that this interval is the same (see also Fig. 3). The flow with $\hat{\mu}_B=0.57$ is the most stable flow (has the highest critical Reynolds numbers).

The critical Reynolds numbers are systematically higher for conducting cylinders with the stabilizing magnetic field [$\hat{\mu}_B$ from interval (19)] and isolating cylinders with a destabilizing magnetic field. The critical Reynolds numbers increase with increasing Hartmann numbers for the stabilizing magnetic field and decrease for the large Ha for the destabilizing magnetic field [26]. The critical Reynolds numbers even vanish if the Hartmann number is larger than some critical Hartmann number, $\mathrm{Ha}_{\mathrm{cr}}$.

Figure 3 presents the critical Hartmann numbers as a function of the axial wave numbers. Note, that for conducting boundary condition the disturbances with smallest $\mathrm{Ha}_{\mathrm{cr}}$ are one dimensional (the axial wave number $k=0$) for all $\hat{\mu}_B$ but $-1<\hat{\mu}_B<0$. The critical Hartmann numbers do not depend on the rotation parameter $\hat{\mu}_\Omega$. Moreover, the axial wave numbers for which critical Reynolds numbers equal zero also do not depend on $\hat{\mu}_\Omega$ (see Fig. 4).

Figure 4 demonstrates that for the unstable rotation ($\hat{\mu}_\Omega < \hat{\eta}^2$) and the unstable magnetic field with $\mathrm{Ha}>\mathrm{Ha}_{\mathrm{cr}}$ the flow is unstable (there is critical Reynolds number) for any wave number which does not equal 0. For stable rotation the flow is unstable only for wave numbers between some minimum wave number $k_{\min}$ and maximum wave number $k_{\max}$ [$(k_{\min}<k_{\max})$]. The $k_{\min}$ decreases and the $k_{\max}$ increases with increasing Ha. So, the interval between critical wave numbers is larger for larger Ha. For conducting boundaries $k_{\min}=0$ except for $-1<\hat{\mu}_B<0$ (see Figs. 3 and 4). This behavior is the same as found by Pessah and Psaltis [7].

Figure 5 presents the dependence of the marginal stability lines on the gap width. We use the negative $\hat{\mu}_B$ value because it is out of the stability interval (19) for any gap width. The critical Hartmann numbers are larger for smaller gap (larger

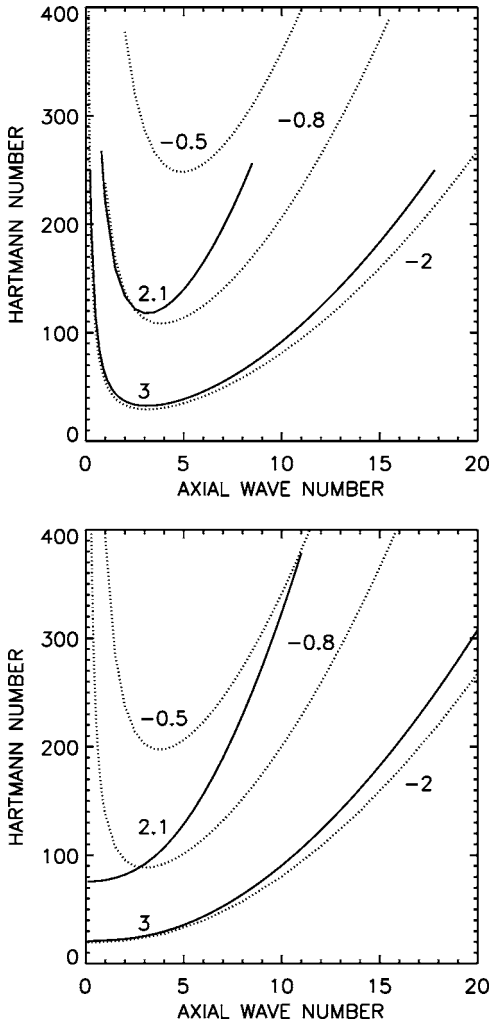

FIG. 3. The marginal stability lines for axisymmetric disturbances ($m=0$) at *insulating* cylinders (upper) and *conducting* ones (lower) with $\hat{\eta}=0.5$ and Re=0. The lines are labeled by the $\hat{\mu}_B$ values.

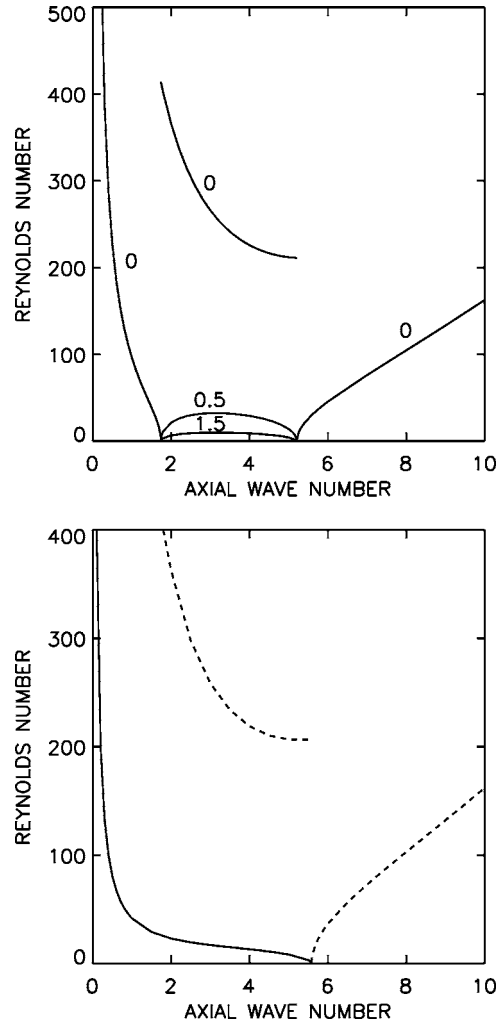

FIG. 4. The marginal stability lines for axisymmetric disturbances ($m=0$) and *insulating* cylinders (upper) and *conducting* ones (lower) with $\hat{\eta}=0.5$, $\hat{\mu}_B=3$, and Ha=40. The lines are labeled by $\hat{\mu}_\Omega$ values on the left panel and $\hat{\mu}_\Omega=0$ (dashed line), 1.5 (solid) on the right panel.

$\hat{\eta}$). For positive $\hat{\mu}_B$ there is some critical gap width. For smaller $\hat{\mu}_B$ than critical gap width, critical Hartmann numbers grow with growing $\hat{\eta}$ the same as for the negative $\hat{\mu}_B$. For larger than critical gap widths, the critical Hartmann numbers grow with decreasing $\hat{\eta}$ due to the approaching of the right boundary of the stability interval (19).

Figure 6 presents the critical Reynolds numbers for nonaxisymmetric disturbances. The results, unlike the axisymmetric case, depend on Pm. Depending on Hartmann numbers the instability is either axisymmetric or asymmetric ($m=1$). Nevertheless, the critical Hartmann number is smaller for $m=1$ mode. The critical Hartmann number does not depend on Pm.

Figure 7 shows the eigenfunctions for the critical Hartmann number (Ha=32.6) for insulating cylinders with $\hat{\mu}_B=3$. The disturbed state has only an azimuthal magnetic field component and has both radial and axial velocities constituting the classical Taylor vortices.

IV. DISCUSSION

The presence of the azimuthal magnetic field can strongly destabilize the Taylor-Couette flow. For the nonideal flow the magnetic field is a fixed function of radius (5) and has only two parameters defined by the flow geometry ($\hat{\eta}$) and the magnetic field boundary values ($\hat{\mu}_B$). The flow can be only destabilized by the magnetic field with $\hat{\mu}_B$ out of range (19). The current free magnetic field ($a_B=0$) has $\hat{\mu}_B=\hat{\eta}$ and stabilizes the flow only in accordance with [21].

The stable magnetic field stabilizes the unstable rotation ($\hat{\mu}_\Omega<\hat{\eta}^2$) and critical Reynolds numbers increase as a function of Hartmann numbers (see Figs. 1 and 2).

The unstable magnetic field possessing the Hartmann number greater than some critical Hartmann number $\mathrm{Ha}_{\mathrm{cr}}$ can destabilize every rotation law. There is instability even without rotation. This instability is the well-known pinch instability [9]. The critical Hartmann number like the critical Reynolds number reflects the nonideality of the flow. The

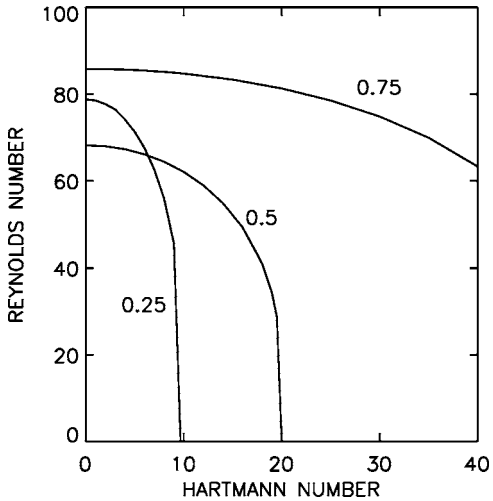

FIG. 5. The marginal stability lines for axisymmetric disturbances ($m=0$) with *conducting* cylinders, Re=0, $\hat{\mu}_\Omega=0$, $\hat{\mu}_B=-2$. The lines are labeled by the $\hat{\eta}$ values.

infinitely small unstable magnetic field destabilizes the cylindrical shell of an ideal fluid without a rotation. The fluid nonideality stabilizes the situation and there is instability only if the magnetic field is large enough ($\mathrm{Ha}>\mathrm{Ha}_{\mathrm{cr}}$).

The stability properties of the flow under the influence of the unstable magnetic field with $\mathrm{Ha}<\mathrm{Ha}_{\mathrm{cr}}$ depend on stability properties of the flow without the magnetic field. The flow, which is stable without the magnetic field ($\hat{\mu}_\Omega>\hat{\eta}^2$), keeps the stability. The critical Reynolds numbers decrease with increasing Hartmann numbers and leads to zero for $\mathrm{Ha}=\mathrm{Ha}_{\mathrm{cr}}$ for the flow, which is unstable without the magnetic field ($\hat{\mu}_\Omega<\hat{\eta}^2$).

The critical Hartmann numbers depend on a geometry (e.g., gap width), the boundary conditions, and the magnetic field profile ($\hat{\mu}_B$) but do not depend on the rotation ($\hat{\mu}_\Omega$).

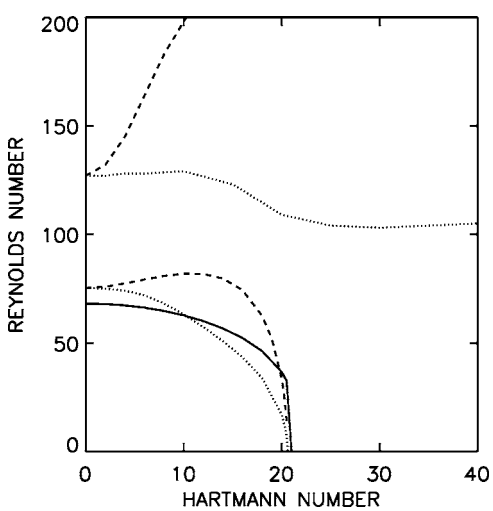

FIG. 6. The marginal stability lines for axisymmetric (*solid*) and nonaxisymmetric disturbances for *conducting* cylinders with $\hat{\eta}=0.5$, $\hat{\mu}_\Omega=0$, $\hat{\mu}_B=3$, and $m=1$ and Pm=1 (*lower dotted*); $m=1$ and $\mathrm{Pm}=10^{-5}$ (*lower dashed*); $m=2$ and Pm=1 (*upper dotted*); $m=2$ and $\mathrm{Pm}=10^{-5}$ (*upper dashed*).

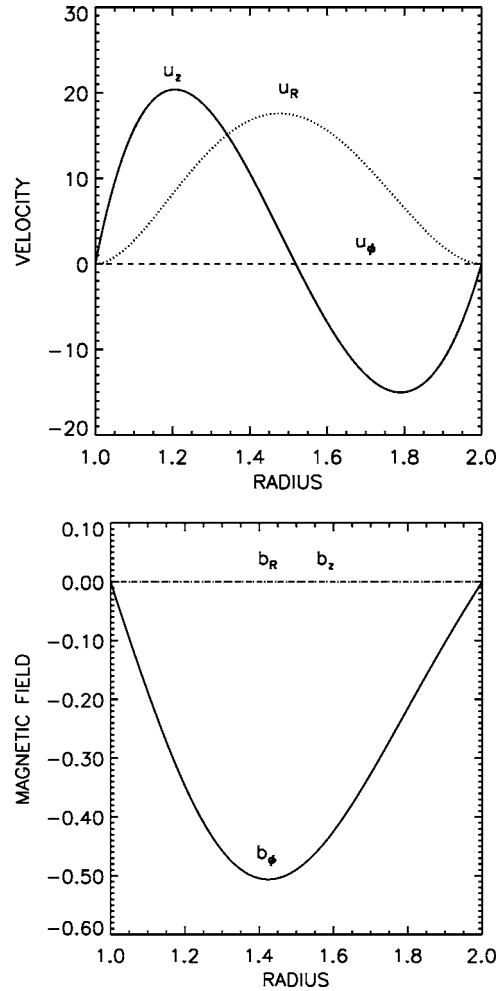

FIG. 7. The eigenfunctions for velocities (upper) and the magnetic field (lower) for *insulating* cylinders and $\hat{\eta}=0.5$, $\hat{\mu}_B=3$, Ha =32.6, and Re=0, $k=1.13$.

Thus, every flow is destabilized by the unstable magnetic field with $\mathrm{Ha}>\mathrm{Ha}_{\mathrm{cr}}$, where $\mathrm{Ha}_{\mathrm{cr}}$ is the same for every $\hat{\mu}_\Omega$ at fixed other parameters. This means that *a rotation cannot stabilize the large enough unstable magnetic field*. The situation is unlike the ideal flow stability. According to (17), the faster the rotation which is stable without magnetic field the larger value of the unstable magnetic field is needed to destabilize the flow.

There are the critical Hartmann numbers for $m=0$ (sausage) and $m=1$ (kink) modes only (see Fig. 6). The sausage mode is the most unstable nearly Ha=0 and the kink mode is the most unstable nearly $\mathrm{Ha}=\mathrm{Ha}_{\mathrm{cr}}$. The transition from the sausage instability to the kink instability depends on Pm. It happens close to $0.5\mathrm{Ha}_{\mathrm{cr}}$ for Pm=1 and almost at $\mathrm{Ha}_{\mathrm{cr}}$ for $\mathrm{Pm}=10^{-5}$. Let us recall that the kink mode is the most unstable mode for the pinch [9]. The critical Hartmann numbers for kink modes are a little bit smaller than for $m=0$ modes (see Fig. 6).

The marginal stability lines for the axisymmetric mode do not depend on the magnetic Prandtl number. The same is true for critical Hartmann numbers for both $m=0$ and $m=1$ modes.

Finally let us estimate the magnetic field needed to achieve zero Reynolds number instability. Taking the parameter values for liquid sodium: $\nu=7.1\times10^{-3}$ cm^2/s, $\eta=810$ cm^2/s, $\rho=0.92$ g/cm^3, and typical dimension $R_{\rm in}=10$ cm, $R_{\rm out}=20$ cm ($\hat{\eta}=0.5$), and $\mathrm{Ha}^2=10^3$ (see Figs. 1 and 2) we get the magnetic field value on the inner cylinder only nearly 30 G and with $\hat{\mu}_B=3$ this corresponds to the 90 G magnetic field on the outer cylinder. The small value of the magnetic field destabilizing the flow and independence of the main results on the magnetic Prandtl number makes the Taylor-Couette flow with imposed azimuthal magnetic field very promising to observe the instability of the magnetize Taylor-Couette flow.

ACKNOWLEDGMENTS

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[24] See Eq. (7) for definitions of $\hat{\mu}_\Omega$ and $\hat{\eta}$.
[25] We are interested in the marginal stability mode for which $\mathcal{I}(\omega)=0$, where $\mathcal{I}(\omega)$ is the imaginary part of ω.
[26] The critical Reynolds numbers can temporarily increase for intermediate Ha as for $\hat{\mu}_B=-0.5$ in Figs. 1 and 2.